Appalachian State University

Department of Mathematics

Sam Auman

Solving Differential Equations Using Symmetry Methods

Solving Differential Equations Using Symmetry Methods

by
Sam Auman

Honors Thesis

Appalachian State University

Submitted to the Department of Mathematical Sciences
in partial fulfillment of the requirements for the degree of
Bachelor of Science

May 2023

Approved by:

____
William J. Cook, Ph.D., Thesis Adviser

____
Quinn A. Morris, Ph.D., Second Reader

____
Nadun Kulasekera Mudiyanselage, Ph.D., Honors Director

____
Eric Marland, Ph.D., Chair, Department of Mathematical Sciences

Abstract

In the late nineteenth century, Sophius Lie developed a technique to solve differential equations using symmetries of solutions. Briefly, a symmetry of a differential equation is a transformation that sends solutions to solutions. In this paper, we explore the connection between a first order differential equation and a corresponding one parameter (local) Lie group of symmetries. Symmetries of a differential equation are then used to find a change of variables (i.e., canonical coordinates) where solving becomes a standard integration problem. We also consider using symmetries to develop an integrating factor which makes our differential equation exact and thus easily solved. Ultimately we leverage symmetries to solve complicated first order differential equations that could not be solved otherwise.

Contents

Introduction

Symmetries are an incredibly useful tool in understanding much of mathematics. As Cantwell says in his text *Introduction to Symmetry Analysis* [C], “Knowledge of these symmetries will be used to enhance our understanding of complex physical phenomena, to simplify and solve problems, and, ultimately, to deepen our understanding of nature.” Sophus Lie and Felix Klein are two of the biggest contributors to the development of applying groups of symmetries to solve problems. In 1870 they worked together trying to understand homogeneous curves, or as they called them, W-curves. These are curves that remained visually the same under a certain group, such as a straight line under the group of translations along the direction of the line or a circle under rotations about the center. Their research together greatly influenced the future of Lie’s research on one-parameter subgroups and the future of Klein’s research relating projective geometry and its group of symmetries. Their relationship was the spark that lead to Klein’s development of symmetry methods in geometry and Lie’s methods for solving differential equations.

We begin by exploring geometrical symmetries on the (x, y)-plane in Section 1.1 and define what is required for a transformation to be considered a symmetry, as well as introduce the concept of one-parameter local Lie groups, which are the symmetries we focus on in this paper. We then explore these concepts in the scope of a homogenous differential equation in Section 1.2 and first order differential equations in Section 2.1. In the following section we look at ODEs with transitional symmetries and find that they can be integrated directly, a result we rely on later in the paper.

To better describe how these Lie groups of symmetries act on solution curves of differential equations, we require understanding of group actions. Section 3.1 delves into this topic along with related concepts such as orbits and stabilizers. With this understanding, Section 3.2 looks at the actions of Lie groups. We begin to explore the concept of invariance, which is when a particular symmetry maps a curve to itself, along with the idea of a characteristic, which helps us determine if a particular symmetry is invariant.

With the above concepts outlined, we finally introduce the notion of canonical coordinates in Chapter 4. We can change coordinates of our first order differential equation to something with translational symmetries, which allow us to solve the given differential equation that we could not ordinarily solve. We go into depth in determining how to find suitable coordinates and carefully develop this method.

A similar method of solving first order differential equations with particular symmetries involves using integrating factors. In our context, such factors are used to make differential equations exact. We are able to define a formula for the integrating factor of a given ODE in Section 5.1, and also work through examples of solving differential equations in this way.

A lot of this paper is drawn from Hydon's *Symmetry Methods for Differential Equations* [H] and greatly expanded upon. Many of the results in his text were given without much explanation of the processes that led to them, so this paper aims to better explain the theory behind this method.

Chapter 1

Introduction to Symmetries

This chapter introduces the concept of symmetry, both on a geometrical object and on a differential equation. Much of the content of this chapter comes from Hydon's *Symmetry Methods for Differential Equations* [H], specifically Sections 1.1 and 1.2.

1.1 Symmetries of Geometrical Objects

"Roughly speaking, a symmetry of a geometrical object is a transformation whose action leaves the object apparently unchanged" [H]. Imagine an equilateral triangle. It can be rotated about the origin by $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, and 2π and visually remain the same. As we can see in Figure 1.1, a rotation maps each vertex to another one. If a symmetry maps each point to itself, that symmetry is called the *trivial symmetry*. Rotating the the equilateral triangle by 2π is a trivial symmetry, as each point on the triangle is mapped to itself.

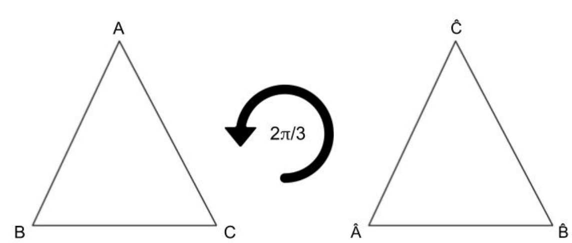


Figure 1.1: Equilateral triangle rotated counterclockwise by $\frac{2\pi}{3}$

Definition 1.1. *A transformation from a set to itself is* **a symmetry** *if it satisfies the following:*

(a) The transformation is invertible.

(b) It preserves our set's structure.

(c) Its inverse also preserves structure.

The symmetries we are concerned with have to follow certain properties. Suppose we have a transformation of the xy-plane, $\Gamma : (x, y) \mapsto (\hat{x}(x, y),\ \hat{y}(x, y))$; this maps a point (x, y) to another point $(\hat{x}, \hat{y})$. If Γ is a symmetry, it must be invertible meaning there exists some transformation Γ^{-1} such that $(\Gamma \circ \Gamma^{-1})(x, y) = (x, y)$ and $(\Gamma^{-1} \circ \Gamma)(x, y) = (x, y)$. For example, if Γ is defined by rotating the plane by $\frac{2\pi}{3}$, then Γ^{-1} is the rotation by $\frac{4\pi}{3}$. Notice this is a geometric symmetry.

To be a symmetry Γ must also be structure preserving. The kind of structure we preserve determines the kind of symmetry we are dealing with. If we are discussing geometric symmetries (i.e., isometries), we preserve distance structure: meaning any two points (x_1, y_1) and (x_2, y_2) stay the same distance apart after being mapped to $(\hat{x}(x_1, y_1),\ \hat{y}(x_1, y_1))$ and $(\hat{x}(x_2, y_2),\ \hat{y}(x_2, y_2))$ respectively.

We will be concerned with symmetries that preserve differentiable structure as we begin to work with differential equations. This then requires our symmetries to be diffeomorphisms. We say $\hat{x}$ and $\hat{y}$ (i.e., the transformation Γ) are smooth if they are infinitely differentiable with respect to x and y (i.e., they have partial derivatives of all orders with respect to all variables). Because we will want the inverse of a symmetry to be a symmetry, we also demand x and y (i.e., Γ^{-1}) to be infinitely differentiable with respect to $\hat{x}$ and $\hat{y}$. Because Γ is a smooth invertible mapping whose inverse is also smooth, Γ is considered a (C^{∞}) *diffeomorphism.*[1]

Defined carefully later, we can show a mapping is a symmetry of a differential equation by showing it is invertible, smooth, sends solutions to solutions, and finally that its inverse is smooth. If a mapping is a symmetry, it also sends solutions back to solutions by the inverse map. We do not need to check for this as it holds automatically.

When using symmetries of a differential equation, we are usually only concerned with *local* symmetries. A local symmetry, Γ, is a diffeomorphism on some neighborhood of a point (x_0, y_0) that sends solutions to solutions. By the inverse function theorem, the inverse Γ^{-1} exists and is differentiable as long as the Jacobian of Γ is invertible at (x_0, y_0). Therefore, to test that Γ is a local symmetry, we need only to check that it is smooth (i.e., its component functions are infinitely differentiable), that the Jacobian matrix at (x_0, y_0) is invertible (i.e., it has a nonzero determinant at (x_0, y_0)), and that Γ maps solutions to solutions. More information about the inverse mapping theorem along with a proof can be found in [E], Theorem 3.3 page 185.

[1] About notation, C^k denotes a collection of maps which have k continuous derivatives. The notation C^{∞} implies we have derivatives of all orders.

If we consider geometric structure and again are dealing with an equilateral triangle, we could count the number of symmetries; three rotations and three reflections. While this had a finite amount of symmetries an object can also have an infinite number of symmetries.

Example 1.2. *The unit circle*

$$x^2 + y^2 = 1$$

has infinitely many geometric symmetries. One such symmetry is defined

$$\Gamma_\varepsilon : (x, y) \to (\hat{x}, \hat{y}) = (x\ \cos(\varepsilon) - y\ \sin(\varepsilon),\ x\ \sin(\varepsilon) + y\ \cos(\varepsilon))$$

for each ε in $(-\pi, \pi]$. When converted into polar coordinates,

$$\Gamma_\varepsilon : (\cos(\theta), \sin(\theta)) \to (\cos(\theta + \varepsilon), \sin(\theta + \varepsilon))$$

which is easy to see that this transformation is a rotation about the origin by any ε in $(-\pi, \pi]$. Rotations about the origin preserve our geometric structure and send the unit circle to itself. Why? Because

$$x^2 + y^2 = 1 \quad \textit{implies} \quad \hat{x}^2 + \hat{y}^2 = 1.$$

The example above turns out to be a one-parameter Lie group.

Definition 1.3. *Suppose we have an infinite set, parameterized by ε, of mappings defined on some subset of $\mathbb{R}^N$. In particular, say of Γ_ε sends $(x^1, x^2, \ldots, x^N)$ to $(\hat{x}^1, \ldots, \hat{x}^N)$ where $\hat{x}^s(x^1, x^2, \ldots, x^N; \varepsilon)$ (i.e., $\hat{x}^s$ is a function of the x^i's along with the indexing variable ε). The inputs $x = (x^1, \ldots, x^N)$ are restriced to some subset of $\mathbb{R}^N$ and ε is a real parameter.*

This set of symmetries is a ***one-parameter (local) Lie group*** *if the following conditions are satisfied:*

(a) Γ_0 is the trivial symmetry, so that $\hat{x}^s(x^1, \ldots, x^N; 0) = x^s$ for each $s = 1, \ldots, N$.

(b) Γ_ε is a symmetry for every ε when restricted to some neighborhood of zero (i.e., open set containing zero).

(c) $\Gamma_\delta \circ \Gamma_\varepsilon = \Gamma_{\delta+\varepsilon}$ for every δ and ε sufficiently close to zero.

(d) Each $\hat{x}^s$ may be represented as a Taylor series in ε (in some neighborhood of $\varepsilon = 0$), and therefore $\hat{x}^s(x^1, ..., x^N; \varepsilon) = x^s + \varepsilon\, \xi^s(x^1, ..., x^N) + O(\varepsilon^2)$ for $s = 1, ..., N$.

For simplicity sake, we will refer to parameterized Lie groups of symmetries as **Lie symmetries**. We shall only be concerned with (local) Lie groups in $\mathbb{R}^2$.

1.2 Symmetries of a Simple ODE

Consider the homogenous ODE

$$\frac{dy}{dx} = 0 \tag{1.1}$$

with a set of solutions on the (x, y)-plane.

Considering Definition 1.1, we want symmetries that preserve our solution structure, so any symmetry must map the solution set to itself. Assume such a symmetry exists and call it $\Gamma : (x, y) \to (\hat{x}(x, y), \hat{y}(x, y))$. Condition (b) requires that Γ maps any solution curve on the (x, y)-plane to another solution curve in the $(\hat{x}, \hat{y})$-plane, thus

$$\frac{d\hat{y}}{d\hat{x}} = 0 \quad \text{when} \quad \frac{dy}{dx} = 0. \tag{1.2}$$

We also want this mapping to be invertible, in other words, we want the determinant of Γ's Jacobian to be nonzero. Thus we need

$$\hat{x}_x \hat{y}_y - \hat{x}_y \hat{y}_x \neq 0$$

to satisfy condition (a) of our definition of a symmetry. We note that the inverse of Γ will automatically send a solution back to a solution. Thus condition (c) of our definition of a symmetry is automatically satisfied.

Using skills learned in an introductory calculus class, it can easily be seen that (1.1) has solutions of the form

$$y(x) = c, \qquad c \in \mathbb{R}. \tag{1.3}$$

We can represent (1.1) geometrically with its set of solutions (1.3) which are horizontal lines that fill the entire (x, y)-plane. A symmetry Γ then maps a particular solution curve of this form to another solution curve, say

$$\hat{y}(x, c) = \hat{c}(c) \quad \text{for all } c \in \mathbb{R} \tag{1.4}$$

where x refers to the inverse function of $\hat{x} = \hat{x}(x, c)$. Because this ODE is so simple, we can find all of its symmetries using its set of solutions. Differentiating (1.4) with respect to x, we obtain:

$$\hat{y}_x(x, c) = 0 \quad \text{for all } c \in \mathbb{R}$$

showing that $\hat{y}$ is a function depending only on y. Thus the symmetries of (1.1) take the form

$$(\hat{x}, \hat{y}) = (f(x, y),\ g(y)), \qquad f_x \neq 0,\ g_y \neq 0 \tag{1.5}$$

where f and g are assumed to be smooth functions.

We can also find these symmetries without already knowing the solution, instead using the condition (1.2). It is safe to say that the solution curves of (1.1) depend on x, so (1.2) can be rewritten as

$$\frac{d\hat{y}}{d\hat{x}} = \frac{D_x\hat{y}}{D_x\hat{x}} = 0 \qquad \text{when} \qquad \frac{dy}{dx} = 0, \tag{1.6}$$

where D_x is the total derivative of x, defined as

$$D_x = \delta_x + y'\delta_y + y''\delta_{y'} + \cdots$$

where δ_x represents $\frac{\partial}{\partial x}$ and so on.

Example 1.4. *Suppose $F = x^2 + xy + y' + (y'')^2$ then we can solve for $D_x(F)$ by recognizing the following equations*

$$\delta_x(F) = 2x + y, \qquad \delta_y(F) = x, \qquad \delta_{y'}(F) = 1$$

$$\delta_{y''}(F) = 2y'' \qquad \delta_{y^{(k)}} = 0 \quad \textit{for all } k > 2$$

therefore $D_x(F) = (2x + y) + xy' + 1 \cdot y'' + 2y'' \cdot y'''$.

With a better understanding of D_x we can expand (1.6)

$$0 = \frac{D_x\hat{y}}{D_x\hat{x}} = \frac{\frac{\delta}{\delta x}(\hat{y}) + y'\frac{\delta}{\delta y}(\hat{y}) + y''\frac{\delta}{\delta y'}(\hat{y}) + \cdots}{\frac{\delta}{\delta x}(\hat{x}) + y'\frac{\delta}{\delta y}(\hat{x}) + y''\frac{\delta}{\delta y'}(\hat{x}) + \cdots} = \frac{\hat{y}_x + y'\hat{y}_y}{\hat{x}_x + y'\hat{x}_y} \quad \text{when} \quad \frac{dy}{dx} = 0. \tag{1.7}$$

Notice that by the nature of (1.1), $y^{(k)} = 0$ for all $k > 1$, thus our expansion has only two terms. However, since $\frac{dy}{dx} = y'$ and according to (1.1) $\frac{dy}{dx} = 0$, we can further simplify (1.7)

$$\frac{\hat{y}_x + y'\hat{y}_y}{\hat{x}_x + y'\hat{x}_y} = \frac{\hat{y}_x + 0 \cdot \hat{y}_y}{\hat{x}_x + 0 \cdot \hat{x}_y} = \frac{\hat{y}_x}{\hat{x}_x} = 0$$

which forces $\hat{y}_x = 0$. Again, this shows that $\hat{y}$ does not depend on x and we can again say that the symmetries of (1.1) take the shape of (1.5).

We can consider (1.2) to be a symmetry condition of (1.1). This condition reveals information about the symmetries without already knowing the solution of the differential equation which is an integral idea for understanding complicated ODEs without a known solution.

Chapter 2

First-Order ODEs

Now that we know what a symmetry is and how it affects a homogenous ODE, we want to establish symmetries of first order differential equations of a certain form. We also aim to find a testable condition to know we have a symmetry. Much of this information is pulled from Hydon's text [H] Sections 1.3 and 1.4.

2.1 The Symmetry Condition

As we saw in the previous section, it is easy to visualize the symmetries of (1.1) because the solution curves are parallel lines. However, most ODEs do not have symmetries that can be easily visualized, so it is important to understand the symmetry conditions. We will be focusing on the next simplest case: A first order differential equation of the form

$$\frac{dy}{dx} = \omega(x, y) \tag{2.1}$$

where ω is a smooth function in some restricted region of the plane. Then, the symmetry condition requires that

$$\frac{d\hat{y}}{d\hat{x}} = \omega(\hat{x}, \hat{y}) \quad \text{when} \quad \frac{dy}{dx} = \omega(x, y). \tag{2.2}$$

We can again use the results from (1.7) as y is a function of x to show

$$\frac{d\hat{y}}{d\hat{x}} = \frac{D_x\hat{y}}{D_x\hat{x}} = \frac{\hat{y}_x + y'\hat{y}_y}{\hat{x}_x + y'\hat{x}_y}$$

and if we substitute in the symmetry condition (2.2), we can define a new, equivalent symmetry condition:

$$\frac{\hat{y}_x + \omega(x, y)\hat{y}_y}{\hat{x}_x + \omega(x, y)\hat{x}_y} = \omega(\hat{x}, \hat{y}) \tag{2.3}$$

given that the mapping, $(x, y) \mapsto (\hat{x}, \hat{y})$, is a diffeomorphism.

Example 2.1. *Consider the ODE*

$$\frac{dy}{dx} = y \tag{2.4}$$

We can use (2.3) where $\omega(x, y) = y$ *to define the symmetry condition for (2.4)*

$$\frac{\hat{y}_x + y \cdot \hat{y}_y}{\hat{x}_x + y \cdot \hat{x}_y} = \hat{y}.$$

Any symmetry of (2.4) must satisfy this partial differential equation.

With the information we currently have, instead of trying to solve this PDE, it is easier to look for certain types of symmetries. We will focus on finding if (2.4) has any symmetries mapping y *to itself. If this is true, then*

$$(\hat{x}, \hat{y}) = (\hat{x}(x, y), y)$$

meaning $\hat{y}_x = 0$ *and* $\hat{y}_y = 1$*. The PDE would then reduce to*

$$\frac{0 + y \cdot 1}{\hat{x}_x + y \cdot \hat{x}_y} = y$$

with the restriction that $\hat{x}_x + y\hat{x}_y = 1$ *and, taking (1.2) into account,* $\hat{x}_x \neq 0$*.*

There are many symmetries that fit these conditions, however one of the simplest is the Lie symmetries

$$(\hat{x}, \hat{y}) = (x + \varepsilon, y), \qquad \varepsilon \in \mathbb{R}.$$

This example better shows the relation between a symmetry condition and the symmetries of an ODE.

Example 2.2. *The Riccati equation*

$$\frac{dy}{dx} = \frac{y+1}{x} + \frac{y^2}{x^3} \tag{2.5}$$

has symmetries of the form

$$(\hat{x}, \hat{y}) = \left(\frac{x}{1 - \varepsilon x}, \frac{y}{a - \varepsilon x} \right). \tag{2.6}$$

We can prove this is true by showing the symmetry condition (2.3) holds. We can assume

$\omega(x,y) = \frac{y+1}{x} + \frac{y^2}{x^3}$ *and from (2.6) we can calculate*

$$\hat{y}_x = \frac{\varepsilon y}{(1-\varepsilon x)^2},$$
$$\hat{y}_y = \frac{1}{1-\varepsilon x},$$
$$\hat{x}_x = \frac{(1-\varepsilon x) - x(-\varepsilon)}{(1-\varepsilon x)^2} = \frac{1}{(1-\varepsilon x)^2},$$
$$\hat{x}_y = 0.$$

Now we can plug all of these values into (2.3) to confirm the symmetry condition stays true with the symmetry (2.6) of (2.5):

$$\begin{aligned}
\frac{\hat{y}_x + \omega(x,y)\hat{y}_y}{\hat{x}_x + \omega(x,y)\hat{x}_y} &= \frac{\frac{\varepsilon y}{(1-\varepsilon x)^2} + \left(\frac{y+1}{x} + \frac{y^2}{x^3}\right)\cdot\left(\frac{1}{1-\varepsilon x}\right)}{\frac{1}{(1-\varepsilon x)^2} + \left(\frac{y+1}{x} + \frac{y^2}{x^3}\right)\cdot 0} \\
&= \frac{\frac{-\varepsilon x^3 + (y+1)x^2 - \varepsilon x y^2 + y^2}{(1-\varepsilon x)^2 x^3}}{\frac{x^3}{(1-\varepsilon)^2 x^3}} \\
&= \frac{-\varepsilon x^3 + (y+1)x^2 - \varepsilon x y^2 + y^2}{x^3} \\
&= \frac{-\varepsilon x^3 + (y+1)x^2 - \varepsilon x y^2 + y^2}{x^3}
\end{aligned}$$

and

$$\begin{aligned}
\omega(\hat{x},\hat{y}) &= \frac{\hat{y}+1}{\hat{x}} + \frac{\hat{y}^2}{\hat{x}^3} \\
&= \frac{(\frac{y}{1-\varepsilon x} + 1)}{\frac{x}{1-\varepsilon x}} + \frac{(\frac{y}{1-\varepsilon x})^2}{(\frac{x}{1-\varepsilon x})^3} \\
&= \frac{(\frac{y}{1-\varepsilon x} + 1)(1-\varepsilon x)x^2 + y^2(1-\varepsilon x)}{x^3} \\
&= \frac{-\varepsilon x^3 + (y+1)x^2 - \varepsilon x y^2 + y^2}{x^3}.
\end{aligned}$$

Because these are equivalent, our symmetry condition holds. Thus the mapping (2.6) is indeed a symmetry.

2.2 Equations with Translational Symmetries

Suppose we have a first order ODE (2.1) that has an identifiable (local) Lie group of symmetries. The general solution can be found from the symmetries, no matter what $\omega(x, y)$ is. To illustrate this result, assume (2.1) has a Lie group of translations defined by

$$(\hat{x}, \hat{y}) = (x, y + \varepsilon) \tag{2.7}$$

which shifts solution curves in the y direction. This translation is a symmetry as long as it meets the symmetry condition (2.3), which in this case reduces to

$$\omega(x, y) = \omega(x, y + \varepsilon) \tag{2.8}$$

for all ε in some neighborhood of 0. Differentiating (2.8) with respect to ε

$$\begin{aligned} \frac{d}{d\varepsilon}(\omega(x, y)) &= \frac{d}{d\varepsilon}(\omega(\hat{x}, \hat{y})) \\ 0 &= \omega_x(x, y + \varepsilon) \cdot \hat{x}_\varepsilon + \omega_y(x, y + \varepsilon) \cdot \hat{y}_\varepsilon \\ 0 &= \omega_x(x, y + \varepsilon) \cdot 0 + \omega_y(x, y + \varepsilon) \cdot 1 \\ 0 &= \omega_y(x, y + \varepsilon). \end{aligned}$$

Evaluated at $\varepsilon = 0$, we get the condition: $\omega_y(x, y) = 0$. Therefore ω does not depend on y. Thus the most general ODE whose symmetries include (2.7) has to be of the form

$$\frac{dy}{dx} = \omega(x)$$

which is very easy to solve. The general solution is

$$y = \int \omega(x) dx + c$$

for any constant c. Notice such a solution maps to

$$\hat{y} = \int \omega(x) dx + c + \varepsilon = \int \omega(\hat{x}) d\hat{x} + c.$$

Thus solution curves are translated to solution curves.

Chapter 3

The Action of Lie Symmetries on the Plane

This chapter introduces the notion of a group action. Given a group, G, and a nonempty set, X, we can define a (left) *group action* that consists of elements of G *acting* on elements of X. We then use this concept to understand how our one-parameter Lie group of symmetries act on solution curves on the (x, y) plane.

3.1 Group Actions

The action of G on X is a map $\cdot : G \times X \to X$ where $(g, x) \mapsto g \cdot x$ for $g \in G$ and $x \in X$. As a reminder, a group has closure, associativity, an identity, and inverses. From this we should want these concepts to hold when acting on X. For closure, given a $g \in G$ and $x \in X$, we need

$$g \cdot x \in X.$$

If we look at associativity, we want for all $g, h \in G$ and $x \in X$ that

$$g \cdot (h \cdot x) = (gh) \cdot x.$$

It is important to note that the left side of this condition has g acting on h acting on x, and on the right we have g multiplied by h acting on x. For the identity we want

$$1 \cdot x = x$$

for all $x \in X$. While the identity acts on the left, it cannot act on the right since only set elements appear on the right. Thus for left group actions, we only have a left identity. Inverses also cause problems, as we cannot have a group element act on the set element and get the group's identity (i.e., a group element) as an output. On the other hand, the statement

$$g \cdot (g^{-1} \cdot x) = (gg^{-1}) \cdot x = 1 \cdot x = x$$

and likewise

$$g^{-1} \cdot (g \cdot x) = x$$

holds.

Given an action, we get a related permutation representation $\varphi : G \to S(X)$ where

$$S(X) = \{f : X \to X \mid f \text{ is invertible}\}$$

or in other words, $S(X)$ is the group of permutations on X, and $\varphi(g) : X \to X$ is the function

$$\varphi(g)(x) = g \cdot x$$

otherwise known as the *action of* g map on X. Notice $\varphi(g)$ is in fact a permutation:

$$(\varphi(g^{-1}) \circ \varphi(g))(x) = g^{-1} \cdot (g \cdot x) = (g^{-1}g) \cdot x = 1 \cdot x = x = \mathrm{Id}_X(x),$$

so $\varphi(g^{-1}) \circ \varphi(g) = \mathrm{Id}_X$ and likewise $\varphi(g) \circ \varphi(g^{-1}) = \mathrm{Id}_X$. So $\varphi(g)$ is indeed invertible and in fact $\varphi(g)^{-1} = \varphi(g^{-1})$. Moreover, φ itself is a homomorphism of groups:

$$\varphi(gh)(x) = (gh) \cdot x = g \cdot (h \cdot x) = \varphi(g)(\varphi(h)(x)) = (\varphi(g) \circ \varphi(h))(x)$$

therefore $\varphi(gh) = \varphi(g) \circ \varphi(h)$ for all $g, h \in G$. As a side note, if $\varphi : G \to S(X)$ is a group homorphism (i.e., a permutation representation), then we could show that $g \cdot x$ defined by $(\varphi(g))(x)$ gives us a G-action on X.

Definition 3.1. *We can define the* orbit *of a point* $x_0 \in X$, $\mathrm{orb}(x_0) = \{g \cdot x_0 \mid g \in G\}$. *Sometimes denoted* $\mathrm{orb}(x_0) = G \cdot x_0$.

To see the orbits partition the set X consider the relation: x_0 and x_1 are related if there is some $g \in G$ such that $g \cdot x_0 = x_1$. For all $x_0 \in X$, we have $1 \cdot x_0 = x_0$ so this relation is reflexive. Consider the points $x_0, x_1 \in X$ such that $x_1 = g \cdot x_0$ for some $g \in G$. Then $g^{-1} \cdot x_1 = g^{-1} \cdot (g \cdot x_0) = (g^{-1}g) \cdot x_0 = 1 \cdot x_0 = x_0$. Ultimately, this means anything related to x_0 is related to x_1 and vice-versa. Thus $\mathrm{orb}(x_0) = \mathrm{orb}(x_1)$ so the relation is also symmetric. Finally, consider a new point $x_2 \in X$ defined by $x_2 = h \cdot x_1$ for some $h \in G$. We can redefine $x_2 = h \cdot (g \cdot x_0) = (hg) \cdot x_0$ which means that $x_0 = (hg)^{-1} \cdot x_0 = g^{-1} \cdot x_1 = 1 \cdot x_0$. Therefore $\mathrm{orb}(x_0) = \mathrm{orb}(x_1) = \mathrm{orb}(x_2)$ thus the relation is also transitive. Now we can see that belonging to the same orbit is an equivalence relation. Also, we see that the orbits are the equivalence classes of this relation.

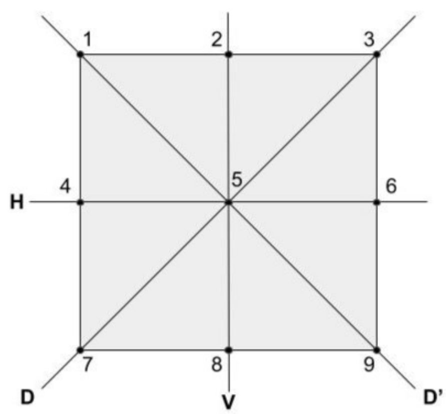
Definition 3.2. *We can define the* stabilizer *of a point* $x_0 \in X$ *as follows:* $\operatorname{stab}(x_0) = \{g \in G \mid g \cdot x_0 = x_0\}$. *This is a subgroup of* G *and is also referred to as an* isotropy group.

Recall that given a subgroup H of a group G, $[G : H]$ is the *index* of H in G. This is the number left (and also right) cosets of H in G.

Theorem 3.3 (The Orbit-Stabilizer Theorem)**.** *Consider a group* G *acting on a set* X. *Then* $[G : \operatorname{stab}(x_0)] = |\operatorname{orb}(x_0)|$. *In particular,* $|\operatorname{orb}(x_0)| \cdot |\operatorname{stab}(x_0)| = |G|$.

This is a well known theorem in mathematics. For a proof see [F] Theorem 16.16 on page 158 of the text.

Example 3.4. *Consider the set of points on a square depicted below and the dihedral group defined by* $D_4 = \{R_{0^\circ}, R_{90^\circ}, R_{180^\circ}, R_{270^\circ}, H, V, D, D'\}$ *where* R_{X° *is a counter-clockwise rotation about* 5 *by* X *degrees and* $H, V, D,$ *and* D' *are reflections across various lines.*



We can have D_4 *act on the set of points* $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ *to demonstrate the ideas we have discussed so far. If we look at* 5*, we can see that no symmetry will move it to another point on the square. Thus we can conclude* $\operatorname{orb}(5) = \{5\}$ *and* $\operatorname{stab}(5) = D_4$. *This is what we would consider an invariant point.*

If we instead looked at 1 *we can see that rotations around the center map* 1 *to the points* $3, 7, 9,$ *and itself. Therefore* $\operatorname{orb}(1) = \{1, 3, 7, 9\}$. *If we want to find the stabilizer of this point, we need to find transformations that act trivially on* 1*. It is easy to see that a reflection across* D' *and a rotation by* $0^\circ = 360^\circ$ *leave* 1 *where it is. Thus* $\operatorname{stab}(1) = \{R_{0^\circ}, D'\}$. *By the Orbit-Stabilizer Theorem, the dimension of* D_4 *will be equivalent to the dimension of* $\operatorname{orb}(1)$ *multiplied by the dimension of* $\operatorname{stab}(1)$*, therefore* $|D_4| = 4 \cdot 2 = 8$. *Notice we could have used the point* 5 *to* *show this result as well, but this would involve us already knowing the size of D_4 as that is the stabilizer for that point.*

We can also note that $\mathrm{orb}(1) = \mathrm{orb}(3) = \mathrm{orb}(7) = \mathrm{orb}(9) = \{1, 3, 7, 9\}$*, which illustrates that belonging to the orbit is indeed an equivalence relation.*

3.2 Lie Symmetry Actions

The information for this section is mostly derived from Hydon's text [H] Section 2.1 starting on page 15.

Suppose that $y = f(x)$ is a solution of the ODE

$$\frac{dy}{dx} = \omega(x,\, y) \tag{3.1}$$

with a symmetry that maps this solution to $\hat{y} = \tilde{f}(\hat{x})$, which is a solution of the ODE

$$\frac{d\hat{y}}{d\hat{x}} = \omega(\hat{x}, \hat{y}).$$

As one might expect, the set of points $(\hat{x}, \hat{y})$ mapped by $y = f(x)$ depend on x and $y = f(x)$. Therefore we can redefine $\hat{x}$ and $\hat{y}$ as follows

$$\hat{x} = \hat{x}(x, f(x)), \qquad \hat{y} = \hat{y}(x, f(x)). \tag{3.2}$$

Because we know this mapping has to be invertible, we can make x a function of $\hat{x}$ and implement it into the second equation of (3.2) to get

$$\hat{y} = \hat{y}(x(\hat{x}), f(x(\hat{x}))) = \tilde{f}(\hat{x}).$$

This tells us that if this symmetry belongs to a one-parameter Lie group, then $\tilde{f}$ is a function of $\hat{x}$ and therefore ε.

Example 3.5. *Consider the ODE*

$$\frac{dx}{dy} = \frac{2y}{x} \tag{3.3}$$

with the general solution $y = cx^2$*. Given the symmetry, a one-parameter Lie group of scalings defined by*

$$(\hat{x}, \hat{y}) = (e^{\varepsilon}x, e^{-\varepsilon}y) \tag{3.4}$$

restricted to $x > 0$ *and* $y > 0$*, what is* $\tilde{f}(\hat{x}, \varepsilon)$*?*

We can see the solution curve when $c = c_1$ *is mapped to*

$$(\hat{x}, \hat{y}) = (e^{\varepsilon}x, c_1 e^{-\varepsilon}x^2).$$

Solving for x as a function depending on $\hat{x}$, it is clear $x = e^{-\varepsilon}\hat{x}$ so then the solution curve must map to

$$\hat{y} = c_1 e^{-3\varepsilon}\hat{x}^2 = \tilde{f}(\hat{x}, \varepsilon).$$

We can superimpose the solution curves of (x, y) and $(\hat{x}, \hat{y})$ in which case the one-parameter Lie group of symmetries can be considered a group action of the symmetries on the (x, y)-plane. Thus a coordinate (x, y) of the solution curve is mapped to the coordinate $(\hat{x}, \hat{y}) = (\hat{x}(x, y),\, \hat{y}(x, y))$ on a possibly different solution curve. In other words, the set of points making up the solution curve $y = f(x)$ is mapped to the set of points making up the solution curve $y = \tilde{f}(x)$. If this mapping does not change anything, meaning $f(x) = \tilde{f}(x)$, then the curve $y = f(x)$ is invariant under the symmetry. A symmetry is considered trivial if its action leaves every solution curve invariant.

Example 3.6. *Again, consider the ODE (3.3). We previously explored the one-parameter Lie group of symmetries (3.4) that map $y = c_1 x^2$ to the curve $\hat{y} = c_1 e^{-3\varepsilon}\hat{x}^2$. This ODE has many other types of symmetries including the discrete symmetry defined by*

$$(\hat{x}, \hat{y}) = \left(\frac{x}{y}, \frac{1}{y}\right).$$

The set of points on the solution curve when $c = c_1$ is mapped to

$$(\hat{x}, \hat{y}) = \left(\frac{x}{c_1 x^2}, \frac{1}{c_1 x^2}\right) = \left(\frac{1}{c_1 x}, \frac{1}{c_1 x^2}\right).$$

Thus the discrete symmetry maps $y = c_1 x^2$ to $\hat{y} = \frac{1}{c_1 x^2} = c_1 \left(\frac{1}{c_1 x}\right)^2 = c_1 \hat{x}^2$ and we can see that this is a trivial symmetry since each solution is mapped back onto itself.

Consider a one-parameter Lie group of symmetries, L, of an ODE (3.1) with the solution $y = f(x)$. We can think of these symmetries as a group acting on points of the (x, y)-plane. Given a point (x, y) on our solution curve $y = f(x)$, the symmetries will act on that point resulting in the points $(\hat{x},\, \hat{y})$ on (potentially) other solution curves. This collection of points is the orbit of the point (x, y) under this group of Lie symmetries.

In particular, given some fixed (x_0, y_0) and one-parameter Lie group, the orbit of (x_0, y_0) is all points

$$(\hat{x},\, \hat{y}) = (\hat{x}(x_0,\, y_0;\, \varepsilon),\, \hat{y}(x_0, y_0;\, \varepsilon)). \tag{3.5}$$

This includes our original point itself:

$$(\hat{x}(x_0, y_0;\, 0),\, \hat{y}(x_0,\, y_0;\, 0)) = (x_0, y_0).$$

So generally the orbit will be a curve of points. If however, (x_0, y_0) only maps to itself, we again call this an *invariant point* for our group of symmetries.

Example 3.7. *Consider the ODE*

$$\frac{dy}{dx} = \frac{y^3 + x^2 y - y - x}{xy^2 + x^3 + y - x}$$

with the one-parameter Lie group of rotational symmetries about the origin defined by

$$(\hat{x}, \hat{y}) = (x\cos(\varepsilon) - y\sin(\varepsilon), x\sin(\varepsilon) + y\cos(\varepsilon)). \tag{3.6}$$

It is very natural to change into polar coordinates for this calculation. Given

$$x = r\ \cos(\theta), \qquad y = r\ \sin(\theta) \tag{3.7}$$

we can greatly simplify the symmetries (3.6) to:

$$(\hat{r}, \hat{\theta}) = (r, \theta + \varepsilon). \tag{3.8}$$

We can test that this is true by the following calculations using (3.7), (3.8), and some trigonometry identities:

$$\begin{aligned}(\hat{x}, \hat{y}) &= (\hat{r}\cos(\hat{\theta}), \hat{r}\sin(\hat{\theta}) \\ &= (r\cos(\theta + \varepsilon), r\sin(\theta + \varepsilon)) \\ &= (r\cos(\theta)\cos(\varepsilon) - r\sin(\theta)\sin(\varepsilon),\ r\cos(\theta)\sin(\varepsilon) + r\sin(\theta)\cos(\varepsilon)) \\ &= (x\cos(\varepsilon) - y\sin(\varepsilon), x\sin(\varepsilon) + y\cos(\varepsilon))\end{aligned}$$

which we can see matches (3.6) as expected. It is pretty trivial to see that an orbit through any point (x_0, y_0) *would be a circle with* $r = \sqrt{{x_0}^2 + {y_0}^2}$*, except if* $(x_0, y_0) = (0, 0)$*. The origin gets mapped to itself and is therefore considered an invariant point.*

Additionally with this change of coordinates we can simplify our ODE (3.7) to:

$$\frac{dr}{d\theta} = r(1 - r^2).$$

This differential equation in these coordinates can now be easily solved using integration.

Consider an orbit through the noninvarient point (x, y). The tangent vector of this orbit at the corresponding point $(\hat{x}, \hat{y})$ can be called $(\xi(\hat{x}, \hat{y}),\ \eta(\hat{x}, \hat{y}))$ defined by

$$\frac{d\hat{x}}{d\varepsilon} = \xi(\hat{x}, \hat{y}), \qquad \frac{d\hat{y}}{d\varepsilon} = \eta(\hat{x}, \hat{y}).$$

This means that the tangent vector at (x, y) is

$$(\xi(x,y),\, \eta(x,y)) = \left(\frac{d\hat{x}}{d\varepsilon}\bigg|_{\varepsilon=0},\, \frac{d\hat{y}}{d\varepsilon}\bigg|_{\varepsilon=0} \right) \tag{3.9}$$

since $\hat{x} = x$ and $\hat{y} = y$ at $\varepsilon = 0$. We can then define the Taylor series for the Lie group action, to first order in ε, as follows:

$$\begin{aligned} \hat{x} &= x + \varepsilon\xi(x,y) + O(\varepsilon^2), \\ \hat{y} &= y + \varepsilon\eta(x,y) + O(\varepsilon^2). \end{aligned} \tag{3.10}$$

As previously defined, an invariant point is mapped to itself, therefore by (3.10), we can see that for this to be true

$$\xi(x,y) = \eta(x,y) = 0. \tag{3.11}$$

In other words, an invariant point has a tangent vector of zero.

We can consider a curve, C, and think of it as a set of points on the (x, y)-plane. If the orbit of a point on C, $(x_0,\, y_0)$, crosses C transversely, this means there are Lie symmetries that map $(x_0,\, y_0)$ to points not on C. Therefore if a solution is invariant, the orbit does not traverse that solution. In other words, for C to be an invariant solution curve we demand the orbit of a point on C to be points on C. Therefore a curve is invariant if and only if the tangent to C at each point (x, y) is parallel to the tangent vector $(\xi(x,y),\, \eta(x,y))$.

We can write out this demand as a mathematical condition. If we consider a curve C: $y = y(x)$, we can see the tangent vector of a point $(x, y(x))$ on C is $(1, y'(x))$. To test if this vector is parallel to the orbit at that point we consider the matrix

$$\begin{bmatrix} 1 & y'(x) \\ \xi(x,y) & \eta(x,y) \end{bmatrix}$$

and its determinant, otherwise known as the *characteristic*:

$$Q(x, y, y') = \eta(x,y) - y'\xi(x,y). \tag{3.12}$$

These tangent vectors are parallel only when (3.12) is equal to zero, making them linearly dependent. Thus $(\xi(x,y), \eta(x,y))$ is parallel to $(1, y'(x))$ if and only if

$$Q(x, y, y') = 0 \qquad \text{on C.}$$

If we again consider the solutions of the ODE (3.1), the characteristic is equivalent to

$$\bar{Q}(x,y) = Q(x, y, \omega(x,y)) = \eta(x,y) - \omega(x,y)\xi(x,y)$$

where we refer to $\bar{Q}(x,y)$ the *reduced characteristic*. We can see a solution curve $y = f(x)$ is thus invariant if and only if

$$\bar{Q}(x,y) = 0 \qquad \text{when} \quad y = f(x). \tag{3.13}$$

To go even further, we observe that the Lie symmetries are trivial if and only if $\bar{Q}(x,y)$ is identically zero, or in other words

$$\eta(x,y) \equiv \omega(x,y)\xi(x,y). \tag{3.14}$$

If $\bar{Q}_y \neq 0$, we know that $\bar{Q}(x,y) = 0$ has a y-dependence. This means we can solve for y and find the invariant solutions $y = y(x)$.

Example 3.8. *Consider the ODE*

$$\frac{dy}{dx} = y$$

which has scaling symmetries of the form

$$(\hat{x}, \hat{y}) = (x, e^{\varepsilon} y). \tag{3.15}$$

We can find the tangent vector of (x,y) by differentiating (3.15) with respect to ε at $\varepsilon = 0$:

$$(\xi(x,y),\, \eta(x,y)) = \left(\frac{d}{d\varepsilon}\Big[x\Big],\, \frac{d}{d\varepsilon}\Big[e^{\varepsilon} y\Big]\right)\Bigg|_{\varepsilon=0} = (0\,,\, e^{\varepsilon} y)\Bigg|_{\varepsilon=0}$$

$$\textit{so that} \qquad (\xi(x,y),\, \eta(x,y)) = (0, y). \tag{3.16}$$

As we identified (3.11), a point is invariant when $\xi(x,y) = \eta(x,y) = 0$, thus the points on $y = 0$ are all invariant.

Example 3.9. *Consider the Riccati equation*

$$y' = xy^2 - \frac{2y}{x} - \frac{1}{x^3}, \qquad (x \neq 0)$$

which has a Lie group of scaling symmetries defined by

$$(\hat{x}, \hat{y}) = (e^{\varepsilon} x, e^{-2\varepsilon} y).$$

We can compute the tangent vector field as defined by (3.9)

$$(\xi(x,y), \eta(x,y)) = (x, -2y).$$

Plugging this in to (3.13) we can see the reduced characteristic is

$$\bar{Q}(x,y) = -2y - \left(xy^2 - \frac{2y}{x} - \frac{1}{x^3}\right)x = \frac{1}{x^2} - x^2y^2.$$

We can see the solution curve of the Riccati equation is invariant only when $\frac{1}{x^2} - x^2y^2 = 0$ *thus there are two invariant solutions:*

$$y = \pm x^{-2}.$$

Thus far we have been using groups of Lie symmetries of an ODE to find the tangent vectors. However, most symmetry methods use the tangent vectors instead of the symmetries themselves. We can use the tangent vectors to reconstruct the Lie symmetries as illustrated in the next example

Example 3.10. *In this example, we aim to reconstruct the Lie symmetries (3.15) from the tangent vector field (3.16). We can substitute (3.16) into (3.9) to get*

$$\frac{d\hat{x}}{d\varepsilon} = 0, \qquad \frac{d\hat{y}}{d\varepsilon} = \hat{y}.$$

The general solution is

$$\hat{x}(x,y;\varepsilon) = A(x,y), \qquad \hat{y}(x,y;\varepsilon) = B(x,y)e^{\varepsilon}.$$

After setting $\varepsilon = 0$ *and using the initial condition (3.5), we find that*

$$(\hat{x},\hat{y}) = (x, e^{\varepsilon}y)$$

as expected.

Chapter 4

Canonical Coordinates

Recall Example 3.7 above along with the topics discussed in Section 2.2. These both illustrate the idea that ODEs (3.1) whose symmetries include the translations:

$$(\hat{x},\, \hat{y}) = (x,\, y + \varepsilon) \tag{4.1}$$

can be integrated directly, which is ideal. Even better, if an ODE after a change of coordinates has a Lie group of symmetries that are equivalent to (4.1), then the ODE can be solved in this coordinate system. We can then convert back to our (x, y) coordinate system to get a solution to the original ODE (3.1). Therefore, we will spend the next section determining how to find these new *canonical coordinates*. The information in this chapter is taken from [H] Sections 2.2 and 2.3 and expanded upon.

We can expect the orbits of the symmetries (4.1) to have the same tangent vector at every point:

$$(\xi(x, y), \eta(x, y)) = (0, 1). \tag{4.2}$$

Thus we want to establish some coordinates

$$(r, s) = r((x, y), s(x, y)) \tag{4.3}$$

such that

$$(\hat{r}, \hat{s}) \equiv (r(\hat{x}, \hat{y}), s(\hat{x}, \hat{y})) = (r, s + \varepsilon) \tag{4.4}$$

given any one-parameter Lie group of symmetries. If we assume this is true, then the tangent vector at the point (r, s) is $(0, 1)$ as determined by:

$$\left.\frac{d\hat{r}}{d\varepsilon}\right|_{\varepsilon=0} = 0, \qquad \left.\frac{d\hat{s}}{d\varepsilon}\right|_{\varepsilon=0} = 1.$$

Using the chain rule, we can rewrite this as

$$\begin{aligned}\left.\frac{d\hat{r}}{d\varepsilon}\right|_{\varepsilon=0} &= \left.(\hat{r}_{\hat{x}} \cdot \hat{x}_\varepsilon + \hat{r}_{\hat{y}} \cdot \hat{y}_\varepsilon)\right|_{\varepsilon=0} \\ \left.\frac{d\hat{s}}{d\varepsilon}\right|_{\varepsilon=0} &= \left.(\hat{s}_{\hat{x}} \cdot \hat{x}_\varepsilon + \hat{s}_{\hat{y}} \cdot \hat{y}_\varepsilon)\right|_{\varepsilon=0}\end{aligned}$$

and by (3.9), we know $\hat{x}_\varepsilon$ and $\hat{y}_\varepsilon$ evaluated at $\varepsilon = 0$ are equivalent to $\xi(x, y)$ and $\eta(x, y)$ respectively. Keeping in mind that $\hat{x} = x$ and $\hat{y} = y$ when $\varepsilon = 0$, we have the condition

$$\begin{aligned}\xi(x, y)r_x + \eta(x, y)r_y = 0 \\ \xi(x, y)s_x + \eta(x, y)s_y = 1.\end{aligned} \tag{4.5}$$

We need the change of coordinates to be invertible (locally). Therefore, we need to introduce the following nondegeneracy condition,

$$r_x s_y - r_y s_x \neq 0. \tag{4.6}$$

Notice that this is nothing more than making sure the Jacobian matrix,

$$J = \frac{\partial(r, s)}{\partial(x, y)} = \begin{bmatrix} r_x & r_y \\ s_x & s_y \end{bmatrix}$$

of our coordinate transformation is invertible. In otherwords, the determinant of J needs to be nonzero. Thus

$$\det(J) = r_x s_y - r_y s_x \neq 0.$$

Any pair of functions $r(x, y)$, $s(x, y)$ satisfying (4.5) and (4.6) is called a pair of *canonical coordinates*.

In the picture below, an arbitrary curve plotted on the (r, s) plane can be seen as well as the curve mapped by a symmetry (4.4). Notice that r is unchanged by this mapping, and is therefore considered an *invariant canonical coordinate*. However s is translated by ε and is not invariant.

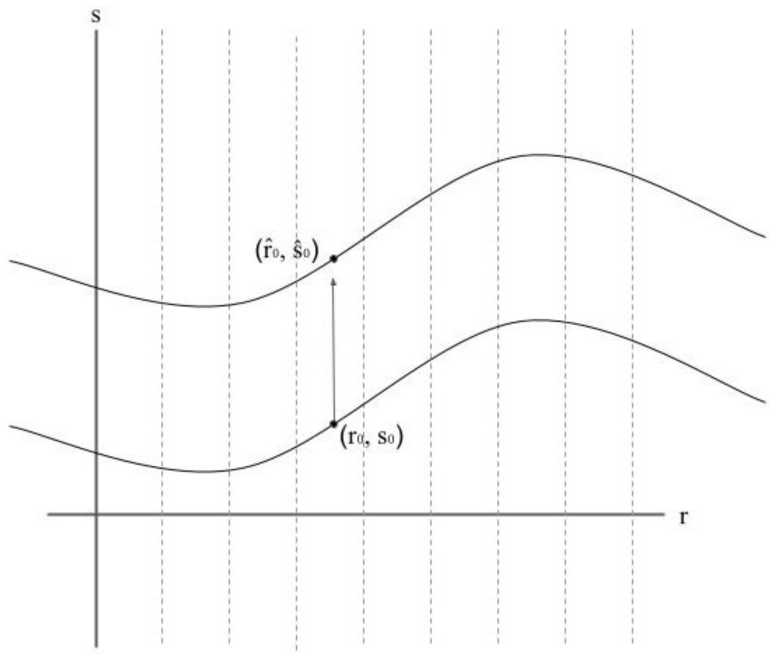


Figure: The canonical coordinate r is invariant, but s is not.

Canonical coordinates cannot be defined at an invariant point, because (3.11) causes the second equation of (4.5) to have no solution. However, canonical coordinates do exist in some neighborhood of any non-invariant point.

Canonical coordinates are not uniquely defined by (4.5). If (r, s) satisfies (4.5), then so does

$$(\tilde{r}, \tilde{s}) = (F(r), s + G(r)),$$

for arbitrary smooth functions F and G such that F is (locally) invertible (i.e., $F'(r) \neq 0$). This means that functions of r and s also work as canonical coordinates. To see this:

$$(\hat{\tilde{r}}, \hat{\tilde{s}}) = (F(r), (s + \varepsilon) + G(r)) = (F(r), s + G(r) + \varepsilon) = (\tilde{r}, \tilde{s} + \varepsilon).$$

Therefore we can see that $\tilde{r}$ is invariant under the symmetry but $\tilde{s}$ gets mapped to $\tilde{s} + \varepsilon$. Then we can translate the nondegeneracy condition (4.6) into these coordinates:

$$\begin{aligned}\tilde{r}_x \tilde{s}_y - \tilde{r}_y \tilde{s}_x &= F'(r) r_x (s_y + G'(r) r_y) - F'(r) r_y (s_x + G'(r) r_x) \\ &= F'(r)(r_x s_y - r_y s_x).\end{aligned}$$

We already know that $r_x s_y - r_y s_x \neq 0$ and $F'(r) \neq 0$ so our nondegeneracy condition still holds.

We intend to rewrite the ODE (3.1) in terms of canonical coordinates. This involves differentiation, so it is wise to use the above freedom to make r and s as simple as possible. For example, it is quite common to find Lie symmetries with η linear in y and ξ independent of y. For these symmetries, if $\xi(x) \neq 0$, there are canonical coordinates with r linear in y and s independent of y. Wherever possible, we shall try to use a simple nondegenerate solution of (4.5).

A *first integral* of a given first-order ODE

$$\frac{dy}{dx} = f(x, y) \tag{4.7}$$

is a nonconstant function $\varphi(x, y)$ whose value is constant on any solution $y = y(x)$ of the ODE (4.7). In other words, if $y = y(x)$ is a solution, $\varphi(x, y(x)) = c$ for some constant c.

Example 4.1. *To understand the concept of a first integral, consider an ODE*

$$\frac{dy}{dx} = f(x, y(x)) \tag{4.8}$$

with the first integral $\varphi(x, y(x)) = c$ *where* $y(x)$ *solves (4.8). We can think of this first integral (set equal to a constant) as an implicit solution for* $\frac{dy}{dx} = f(x, y)$. *Consider taking the derivative of this first integral:*

$$\varphi_x(x, y(x)) + \varphi_{y(x)}(x, y(x)) \cdot y'(x) = 0$$

and because $y(x)$ *solves (4.8), we can replace* $y'(x) = f(x, y(x))$ *to get our condition:*

$$\varphi_x(x, y(x)) + \varphi_{y(x)}(x, y(x)) \cdot f(x, y(x)) = 0.$$

Now, we can see that $\varphi(x, y(x)) = c$ is an implicit solution of (4.7) such that

$$\varphi_x + f(x, y)\varphi_y = 0, \qquad \varphi_y \neq 0. \tag{4.9}$$

Suppose that $\xi(x, y) \neq 0$. If we look at the first equation of (4.5) we can divide by $\xi(x, y)$ to get:

$$r_x + \frac{\eta(x, y)}{\xi(x, y)} r_y = 0$$

which when compared to (4.9) tells us that the invariant canonical coordinate r is a first integral of

$$\frac{dy}{dx} = \frac{\eta(x, y)}{\xi(x, y)}. \tag{4.10}$$

Therefore $r = \varphi(x, y)$ is found by solving (4.10).

Canonical coordinates can be obtained from (4.5) by using the method of characteristics. Assuming that s is independent of y we know that the second line of (4.5) simplifies to

$$\xi(x, y) \cdot s_x + \eta(x, y) \cdot 0 = 1$$

which is equivalent to

$$\xi(x,y)\frac{ds}{dx} = 1$$

therefore

$$ds = \frac{dx}{\xi(x,y)}$$

again assuming that $\xi(x,y) \neq 0$. From (4.10) we can see that

$$\frac{dx}{\xi(x,y)} = \frac{dy}{\eta(x,y)}.$$

Therefore we can define the characteristic equations to be

$$\frac{dx}{\xi(x,y)} = \frac{dy}{\eta(x,y)} = ds. \tag{4.11}$$

Quite often, a solution $s(x,y)$ of (4.5) may be found by inspection. Otherwise we can use $r = r(x,y)$ to write y as a function of r and x. Then the coordinate $s(r,x)$ is obtained from (4.11) by quadrature,

$$s(r,x) = \left(\int \frac{dx}{\xi(x,y(r,x))}\right)\bigg|_{r=r(x,y)}; \tag{4.12}$$

here the integral is evaluated with r being treated as a constant.

Similarly, if $\xi(x,y) = 0$ and $\eta(x,y) \neq 0$ then we can safely choose $r = x$, giving

$$s = \left(\int \frac{dy}{\eta(r,y)}\right)\bigg|_{r=x}.$$

Example 4.2. *Consider the following Lie symmetries:*

$$(\hat{x}, \hat{y}) = (e^{\varepsilon}x, e^{k\varepsilon}y), \qquad k > 0.$$

We can find the tangent vector to be

$$(\xi(x,y), \eta(x,y)) = (x, ky),$$

therefore, by (4.10), r is the first integral of

$$\frac{dy}{dx} = \frac{ky}{x}.$$

Since this can be integrated directly, we can see that the solution of this ODE is $y = cx^k$ thus $x^{-k}y = c$, so we have $r = x^{-k}y$. For $x \neq 0$, ξ is nonzero and does not depend on y, so we can *use (4.12) to find s as follows:*

$$\begin{aligned} s(r,x) &= \int \frac{dx}{\xi(x,y(r,x))} \Bigg|_{r=r(x,y)} \\ &= \int \frac{1}{x}\,dx \Bigg|_{r=x^{-k}y} \\ &= \ln|x| \Bigg|_{r=x^{-k}y} \\ &= \ln|x| \end{aligned}$$

Therefore

$$(r,s) = (x^{-k}y, \ln|x|)$$

when $x \neq 0$. Notice these canonical coordinates cannot be used on the whole plane. More specifically, they fail on the line $x = 0$. To find coordinates that can be used near $x = 0$ we first need to choose a different formula for r. Recall that we have some flexibility in our choice. We can replace r with $F(r) = 1/r$ and get our new: $r = x^k y^{-1}$. Then we can use (4.11) to find s:

$$\begin{aligned} s(x,y) &= \int \frac{1}{\eta(x,y)}\,dy \\ &= \int \frac{1}{ky}\,dy \\ &= \frac{1}{k}\ln|y| \end{aligned}$$

However, we can see that these canonical coordinates do not work around $y = 0$. The only remaining uncovered point is $(x,y) = (0,0)$. However, canonical coordinates do not exist at invariant points (i.e., at $(0,0)$).

As we can see in the above example, because canonical coordinates cannot be defined at an invariant point it is often times neccessary to find coordinates to patch up the missing sections where canonical coordinates can exist.

Example 4.3. *Consider the one-parameter Lie group of inversions*

$$(\hat{x},\hat{y}) = \left(\frac{x}{1-\varepsilon x}, \frac{y}{1-\varepsilon x}\right). \tag{4.13}$$

Again, we can compute the tangent vector to be

$$(\xi(x,y), \eta(x,y)) = (x^2, xy)$$

thus r is a solution of the ODE

$$\frac{dy}{dx} = \frac{y}{x}.$$

This equation can be integrated directly and has the solution $y = cx$, thus we choose $r = \frac{y}{x}$. We then can solve for s again using (4.12) as follows:

$$\begin{aligned} s(r,x) &= \left(\int \frac{dx}{\xi(x, y(r,x))}\right)\Big|_{r=r(x,y)} \\ &= \left(\int \frac{1}{x^2} dx\right)\Big|_{r=\frac{y}{x}} \\ &= \left(-\frac{1}{x}\right)\Big|_{r=\frac{y}{x}} \\ &= -\frac{1}{x} \end{aligned}$$

therefore

$$(r,s) = \left(\frac{y}{x}, -\frac{1}{x}\right). \tag{4.14}$$

Every point on the line $x = 0$ is invariant, so we cannot define canonical coordinates there.

We will use canonical coordinates to find solutions of ODEs. However, we can also easily reconstruct Lie symmetries from canonical coordinates as follows: First, we need to write x and y in terms of r and s:

$$x = f(r,s), \qquad y = g(r,s).$$

Then using (4.4) we can define the symmetries

$$\begin{aligned} \hat{x} &= f(\hat{r}, \hat{s}) = f(r(x,y),\, s(x,y) + \varepsilon), \\ \hat{y} &= g(\hat{r}, \hat{s}) = g(r(x,y),\, s(x,y) + \varepsilon). \end{aligned} \tag{4.15}$$

Example 4.4. *Consider the tangent vector*

$$(\hat{x}, \hat{y}) = \left(\frac{x}{1-\varepsilon x}, \frac{y}{1-\varepsilon x}\right)$$

as seen in Example 4.3 with the canonical coordinates defined as

$$(r,s) = \left(\frac{y}{x}, -\frac{1}{x}\right).$$

To reconstruct the symmetries from the canonical coordinates, we solve these coordinates for x and y:

$$s = -\frac{1}{x} \implies x = -\frac{1}{s}, \qquad \text{and} \qquad r = \frac{y}{x} = \frac{y}{-\frac{1}{s}} \implies y = -\frac{r}{s}$$

therefore

$$(x,y) = \left(-\frac{1}{s}, -\frac{r}{s}\right).$$

By (4.15)

$$(\hat{x}, \hat{y}) = \left(-\frac{1}{s+\varepsilon}, -\frac{r}{s+\varepsilon}\right) = \left(-\frac{1}{-\frac{1}{x}+\varepsilon}, -\frac{\frac{y}{x}}{-\frac{1}{x}+\varepsilon}\right) = \left(\frac{x}{1-\varepsilon x}, \frac{y}{1-\varepsilon x}\right)$$

which matches (4.13) as expected.

4.1 How to Solve ODEs with Lie Symmetries

Now that we known how to find canonical coordinates, we need to learn how to use them to solve an ODE (3.1). Suppose we have identified nontrivial Lie symmetries of an ODE of this form. As a reminder, we found in Section 3.2 (see 3.14) that Lie symmetries are nontrivial if and only if

$$\eta(x, y) \not\equiv \omega(x, y)\xi(x, y).$$

We can then rewrite the ODE (3.1) in terms of canonical coordinates. Recall in Section 2.1 we defined (2.3) to test our symmetry. If we imagine this change of coordinates as a symmetry such that $(\hat{x}, \hat{y}) = (r, s)$ we can use this result to change to canonical coordinates as follows:

$$\frac{ds}{dr} = \frac{s_x + \omega(x, y)s_y}{r_x + \omega(x, y)r_y}. \tag{4.16}$$

This gives us a function of (x, y) but we can change coordinates of the right-hand side of (4.16) to get a function of r and s. We can then define this for a general change of variables $(x, y) \mapsto (r, s)$ as

$$\frac{ds}{dr} = \Omega(r, s), \tag{4.17}$$

for some function Ω. However we have defined (r, s) to be canonical coordinates, and so the ODE is invariant under the group of translations in the s direction:

$$(\hat{r}, \hat{s}) = (r, s + \varepsilon).$$

Therefore, from Section 2.2, the ODE (4.17) does not depend on s and can be represented as

$$\frac{ds}{dr} = \Omega(r). \tag{4.18}$$

The problem is now reduced to integration. We can integrate both sides of (4.18) in terms of r which results in

$$\int \frac{ds}{dr} dr = \int \Omega(r) dr.$$

This gives a general solution

$$s - \int \Omega(r) dr = c,$$

where c is an arbitrary constant. Therefore we can define the general solution of the ODE (3.1) to be

$$s(x,y) - \int^{r(x,y)} \Omega(r)dr = c. \tag{4.19}$$

As long as we can define a non-trivial one-parameter Lie group of symmetries, we can determine canonical coordinates and apply this method to solve any ODE (3.1). This is an incredibly powerful tool to simplify complicated ODEs and even solve them by hand.

Example 4.5. *Recall the Riccati equation*

$$y' = xy^2 - \frac{2y}{x} - \frac{1}{x^3}, \qquad (x \neq 0), \tag{4.20}$$

that we found in Example 3.9 to be invariant under the Lie symmetries

$$(\hat{x}, \hat{y}) = (e^{\varepsilon}x, e^{-2\varepsilon}y).$$

From Example 4.2 we know that the canonical coordinates for symmetries of this form are

$$(r, s) = (x^2y, \ln|x|).$$

With this we can compute $\frac{ds}{dr}$ *from (4.16) as follows*

$$\begin{aligned}\frac{ds}{dr} &= \frac{\frac{1}{x} + 0}{2xy + (xy^2 - \frac{2y}{x} - \frac{1}{x^3})x^2} \\ &= \frac{\frac{1}{x}}{2xy + x^3y^2 - 2yx - \frac{1}{x}} \\ &= \frac{1}{x^4y^2 - 1}\end{aligned}$$

which in terms of r *and* s *simplifies to*

$$\frac{ds}{dr} = \frac{1}{r^2 - 1}.$$

Now we can calculate the solution of this new ODE from (4.19) to be

$$
\begin{aligned}
c &= s(x,y) - \int^{r(x,y)} \Omega(r)dr \\
&= \ln|x| - \int^{x^2y} \frac{1}{r^2-1}dr \\
&= \ln|x| - \left[\frac{1}{2}\ln|r-1| - \frac{1}{2}\ln|r+1|\right]\Bigg|_{x^2y} \\
&= \ln|x| - \left[\ln\sqrt{\left|\frac{r-1}{r+1}\right|}\right]\Bigg|_{x^2y} \\
&= \ln|x| - \ln\sqrt{\left|\frac{x^2y-1}{x^2y+1}\right|}.
\end{aligned}
$$

Finally we can solve for y to find the general solution of (4.20). Note that we freely rename our constant c to simplify notation.

$$
\begin{aligned}
c &= \ln|x| - \ln\sqrt{\left|\frac{x^2y-1}{x^2y+1}\right|} \\
c &= \ln\left(\frac{x}{\sqrt{\left|\frac{x^2y-1}{x^2y+1}\right|}}\right) \\
c &= \frac{x}{\sqrt{\left|\frac{x^2y-1}{x^2y+1}\right|}} \\
c &= \frac{x^2}{\frac{x^2y-1}{x^2y+1}} \\
c &= \frac{x^2(x^2y+1)}{x^2y-1} \\
cx^2y - c &= x^4y + x^2 \\
cx^2y - x^4y &= x^2 + c \\
y &= \frac{x^2+c}{x^2(c-x^2)}
\end{aligned}
$$

so we have found the general solution of (4.20) to be:

$$
y = \frac{x^2+c}{x^2(c-x^2)}. \tag{4.21}
$$

The invariant solution curve $y = x^{-2}$ can be regarded as the limit of (4.21) as c approaches infinity. The other invariant solution is obtained by setting $c = 0$ in (4.21).

Example 4.6. *In Example 2.2, we found that the ODE*

$$\frac{dy}{dx} = \frac{y+1}{x} + \frac{y^2}{x^3} \tag{4.22}$$

has Lie symmetries of the form

$$(\hat{x}, \hat{y}) = \left(\frac{x}{1 - \varepsilon x}, \frac{y}{1 - \varepsilon x} \right).$$

We found the canonical coordinates for symmetries of this form in Example 4.3 to be (4.14):

$$(r, s) = \left(\frac{y}{x}, -\frac{1}{x} \right) \qquad x \neq 0.$$

We can again use (4.16) to find the ODE (4.22) in terms of r and s:

$$\begin{aligned} \frac{ds}{dr} &= \frac{\frac{1}{x^2} + 0}{-\frac{y}{x^2} + \left(\frac{y+1}{x} + \frac{y^2}{x^3} \right) \frac{1}{x}} \\ &= \frac{\frac{1}{x^2}}{-\frac{y}{x^2} + \frac{y}{x^2} + \frac{1}{x^2} + \frac{y^2}{x^4}} \\ &= \frac{1}{1 + (\frac{y}{x})^2} \\ &= \frac{1}{1 + r^2}. \end{aligned}$$

Using (4.19) we can find the general solution to be:

$$c = s - \arctan(r),$$

which when plugging in $r = \frac{y}{x}$ and $s = -\frac{1}{x}$ is:

$$c = -\left(\frac{1}{x} + \arctan\left(\frac{y}{x} \right) \right)$$

which can then be rearranged into the final solution of the ODE (4.22):

$$y = -x \tan\left(\frac{1}{x} + c \right).$$

Example 4.7. *Consider the ODE*

$$y' = \frac{y - 4xy^2 - 16x^3}{y^3 + 4x^2y + x} \tag{4.23}$$

which has Lie symmetries whose tangent vector field is

$$(\xi(x, y), \eta(x, y)) = (-y, 4x).$$

We can find the characteristic equation for r *to be:*

$$\frac{dy}{dx} = -\frac{4x}{y},$$

as determined by (4.10) in the previous section. This has the general solution $y^2 = -4x^2 + c$ *so we can choose* $r = \sqrt{4x^2 + y^2}$.

Consider the region $y > 0$. *In this region* $y(r, x) = \sqrt{r^2 - 4x^2}$, *so our tangent vector* $\xi(x, y(r, x)) = -\sqrt{r^2 - 4x^2}$. *Therefore, using (4.12) we can find a second canonical coordinate to be*

$$s = -\int^x \frac{dx}{\sqrt{r^2 - 4x^2}} = \frac{1}{2}\operatorname{arccos}\left(\frac{2x}{r}\right) = \frac{1}{2}\operatorname{arccot}\left(\frac{2x}{y}\right), \qquad s \in (0, \pi/2)$$

where we can swap out $\operatorname{arccos}\left(\frac{2x}{r}\right)$ *with* $\operatorname{arccot}\left(\frac{2x}{y}\right)$ *since if a triangle has adjacent side* $2x$ *and hypotenuse* r, *then its opposite side would be* $\sqrt{r^2 - 4x^2} = \sqrt{4x^2 + y^2 - 4x^2} = y$. *Note that we need to keep* $y > 0$ *and* $x \neq 0$.

In this region, the ODE (4.23) reduces to

$$\frac{ds}{dr} = -r.$$

The solution to this ODE is

$$c = s + \frac{r^2}{2}$$

and after reverting to the original variables, we obtain an implicit solution of (4.23):

$$y\cos(4x^2 + y^2 + c) + 2x\sin(4x^2 + y^2 + c) = 0.$$

4.2 Trivial Symmetries Do Not Help

Thus far we have required that the Lie symmetries must be nontrivial. We might ask what happens if we encounter trivial symmetries. Suppose that a given ODE (3.1) has a trivial one-parameter Lie group of symmetries in which (r, s) are canonical coordinates. We will represent the general solution of the ODE as

$$\varphi(r, s) = c.$$

Every solution is invariant under the action of the Lie symmetries, and therefore

$$\varphi(r, s + \varepsilon) = \varphi(r, s),$$

for all ε sufficiently close to zero. Hence φ is independent of s andwe can write the general solution of the ODE as

$$r = c.$$

This means we only need to find r, which is a first integral of

$$\frac{dy}{dx} = \frac{\eta(x, y)}{\xi(x, y)}. \tag{4.24}$$

Remember that Lie symmetries of (3.1) are trivial if and only if

$$\eta(x, y) \equiv \omega(x, y)\xi(x, y),$$

and so (4.24) reduces to the original ODE (3.1) with our canonical coordinates of a trivial Lie group of symmetries. Because the purpose of this method is to simplify ODEs into something more easily solved, these coordinates are useless. This is why we have required that the Lie group of symmetries we base our canonical coordinates from must be nontrivial.

Chapter 5

Equations with Particular Symmetries

We have used canonical coordinates to solve ODEs of the form (3.1) thus far. Now we shift to a second technique using integrating factors to solve ODEs. In this scenario, Lie symmetries tell us how to find the necessary integrating factor.

Suppose we now have an ODE of the form

$$\frac{dy}{dx} = \frac{B(x,y)}{A(x,y)} \tag{5.1}$$

which we can see is equivalent to

$$-Bdx + Ady = 0. \tag{5.2}$$

We call this form of the ODE its *Pfaffian form*. To be more specific, technically (5.2) is a differential form. However, we are able to manipulate this equation without needing these formalities, so we will not go into details about this kind of form.

Definition 5.1. *A first-order differential equation of the form*

$$P(x,y)dy + Q(x,y)dx = 0 \qquad \textit{or equivalently} \qquad P(x,y)\frac{dy}{dx} + Q(x,y) = 0$$

is considered exact *if*

$$\frac{\partial P(x,y)}{\partial x} = \frac{\partial Q(x,y)}{\partial y}.$$

Exact differential equations can be easily solved. The following example illustrates this and helps form an idea as to why this is true.

Example 5.2. *Consider the differential equation*

$$(2xy+1)y' + (y^2 + e^x) = 0. \tag{5.3}$$

We can see that for this differential equation, $P = 2xy + 1$ *and* $Q = y^2 + e^x$ *in the second equation of Definition 5.1. To see if this is indeed an exact differential equation, we need to test if* $\frac{\partial P(x,y)}{\partial x} = \frac{\partial Q(x,y)}{\partial y}$*. We can compute*

$$\frac{\partial P(x,y)}{\partial x} = 2y \qquad \textit{and} \qquad \frac{\partial Q(x,y)}{\partial y} = 2y$$

so our equation is indeed exact! It turns out that, given an arbitrary constant c, $c = \int P\,dy = \int Q\,dx$ *(each integral undoing partial differentiation) is an implicit solution of our ODE. We integrate:*

$$\int (2xy + 1)dy = xy^2 + y + C_1(x) \tag{5.4}$$

where $C_1(x)$ *accounts for an arbitrary function of* x*. Similarly*

$$\int (y^2 + e^x)dx = xy^2 + e^x + C_2(y) \tag{5.5}$$

where $C_2(y)$ *account for an arbitrary function of* y*. We can see that* xy^2 *appears in both (5.4) and (5.5), so we need only one copy in the implicit solution. Additionally, we can conclude that* $C_1(x) = e^x$ *and* $C_2(y) = y$ *plus some constant when comparing (5.4) and (5.5). These must also both show up once in the implicit solution. Therefore, after setting the equation equal to a constant, we get that an general solution of the ODE (5.3) is:*

$$xy^2 + y + e^x = c$$

which gives the implicit solution

$$xy^2 + y + e^x - c = 0.$$

Because exact differential equations are so nice to work with, we want our ODE (5.2) to be exact. Therefore we need $-B_y = A_x$ to be true. We want to introduce an *integrating factor* that forces our equation to be exact so we can solve our ODE. More information on integrating factors can be found in [I] in Section 2.1.5 starting on page 32 or [C] in Section 3.3.2 starting on page 59.

5.1 Finding an Integrating Factor

Recall in Chapter 4 that we established the concept of a first integral. Consider the first integral

$$\varphi(x,y) = c$$

of the ODE (5.1) where c is a constant depending on the initial condition. In Example 4.1 we found that the derivative of this first integral is

$$\varphi_x dx + \varphi_y dy = 0 \tag{5.6}$$

where dx can be thought of as the derivative of x and likewise dy the derivative of y. If we suppose $y = y(x)$ is an explicit solution then we can further simplify (5.6) to

$$\begin{aligned} 0 &= \varphi_x \tfrac{dx}{dx} + \varphi_y \tfrac{dy}{dx} \\ &= \varphi_x + \varphi_y \tfrac{dy}{dx}. \end{aligned}$$

Finally, we can implement (5.1) into this equation to get

$$\varphi_x(x, y) + \varphi_y(x, y)\,\frac{B(x, y)}{A(x, y)} = 0$$

or

$$A(x, y)\varphi_x(x, y) + B(x, y)\varphi_y(x, y) = 0.$$

When finding our integrating factor we need this to be true.

Our Lie group of symmetries still send solutions to solutions by mapping points (x, y) to $(\hat{x}, \hat{y})$ where $\hat{x}$ and $\hat{y}$ still depend on ε. Therefore because $\varphi(x, y) = c$ is a solution $\varphi(\hat{x}, \hat{y}) = c(\varepsilon)$ must also be a solution depending on ε. Recall that $\varepsilon = 0$ refers to the trivial symmetry, thus $c = c(0)$.

If we expand in terms of this solution's Maclaurin series (in ε), we see the first term is

$$\varphi(\hat{x}, \hat{y})\Big|_{\varepsilon=0} = \varphi(x, y).$$

The second term of the Maclaurin series is

$$\left((\varphi_{\hat{x}}(\hat{x}, \hat{y}) \cdot \hat{x}_\varepsilon)\Big|_{\varepsilon=0} + (\varphi_{\hat{y}}(\hat{x}, \hat{y}) \cdot \hat{y}_\varepsilon)\Big|_{\varepsilon=0}\right) \cdot \varepsilon = (\varphi_x(x, y) \cdot \xi(x, y) + \varphi_y(x, y) \cdot \eta(x, y)) \cdot \varepsilon$$

and we can refer to the rest of the terms as $O(\varepsilon^2)$. For all points (x, y) lying on the solution curve $\varphi(x, y) = c(0)$ is a solution, therefore $\varphi_x \cdot \xi + \varphi_y \cdot \eta = c'(0)$ must be true where $c'(0)$ is some constant depending on ε. Without loss of generality, we can change the parameter ε in $(\hat{x}, \hat{y})$ to some scaled parameter, say $k\varepsilon$, such that $c'(0) = 1$. This gives us another condition for the integrating factor:

$$\varphi_x \cdot \xi + \varphi_y \cdot \eta = 1.$$

To find exactly what φ_x and φ_y are, we need to use the following defined method from Linear Alegbra:

Definition 5.3. (Cramer's Rule) *Let $Ax = b$ be the matrix form of a system of n linear equations in n unknowns, where $x = [x_1\ x_2\ \cdots\ x_n]^T$. If $\det(A) \neq 0$, then this system has a unique solution, and for each k ($k = 1, 2, \ldots, n$),*

$$x_k = \frac{\det(M_k)}{\det(A)},$$

where M_k is the $n \times n$ matrix obtained from A by replacing column k of A by b.

A proof of this result as well as more information can be found in [FIS] Section 4.3, Theorem 4.9 on page 224. Now we can use Cramer's rule as follows:

$$\varphi_x = \det \begin{bmatrix} 0 & B \\ 1 & \eta \end{bmatrix} \Big/ \det \begin{bmatrix} A & B \\ \xi & \eta \end{bmatrix} = \frac{-B}{A\eta - B\xi} = \frac{1}{A\eta - B\xi} \cdot (-B)$$

and

$$\varphi_y = \det \begin{bmatrix} A & 0 \\ \xi & 1 \end{bmatrix} \Big/ \det \begin{bmatrix} A & B \\ \xi & \eta \end{bmatrix} = \frac{A}{A\eta - B\xi} = \frac{1}{A\eta - B\xi} \cdot A.$$

Therefore we can finally define our integrating factor to be

$$M = \frac{1}{A\eta - B\xi}. \tag{5.7}$$

In particular, $M(-Bdx + Ady) = M \cdot 0$ is equivalent to $\varphi_x dx + \varphi_y dy = 0$ as we would hope. In other words, multiplying our equation by the integrating factor M makes it into an exact equation.

We can solve our original equation (5.1) from (5.2) using

$$\int M(-B)dx = \int M(A)dy = c$$

where c is some constant. Keep in mind, just like in Example 5.2, if a term shows up in both $\int M(-B)dx$ and $\int M(A)dy$, only one copy of that term appears in the implicit solution.

Depending on the ODE, we have different symmetries yielding different tangent vectors $(\xi(x, y), \eta(x, y))$. The table below from [C], Section 6.4 page 158, provides a collection of general formulas for ODEs corresponding to various commonly occurring symmetries:

Equation	ξ	η	Equation	ξ	η
$y' = F[y]$	1	0	$y' = \frac{y}{x} + xF\left[\frac{y}{x}\right]$	1	$\frac{y}{x}$
$y' = F[x]$	0	1	$xy' = y + F\left[\frac{y}{x}\right]$	x^2	xy
$y' = F[ax + by]$	b	$-a$	$y' = \frac{y}{x + F[\frac{y}{x}]}$	xy	y^2
$y' = \frac{y + xF[x^2 + y^2]}{x - yF[x^2 + y^2]}$	y	$-x$	$y' = \frac{y}{x + F[y]}$	y	0
$y' = F\left[\frac{y}{x}\right]$	x	y	$xy' = y + F[y]$	0	x
$y' = x^{k-1}F\left[\frac{y}{x^k}\right]$	x	ky	$xy' = \frac{y}{\ln[x] + F[y]}$	xy	0
$xy' = F[xe^{-y}]$	x	1	$xy' = y(\ln[y] + F[x])$	0	xy
$y' = yF[ye^{-x}]$	1	y	$y' = yF[x]$	0	y

where F is an arbitrary function.

Notice $\xi = 0$ and $\eta = 1$ is the vertical translation symmetry that we saw in Section 3.2 in Example 3.8. We saw that the tangent vector of the symmetry of this equation was exactly

$$(\xi(x, y), \eta(x, y)) = (0, y)$$

just as the table predicts. The symmetry corresponding to $\xi = 1$ and $\eta = 0$ is that of horizontal translational symmetry, which is the symmetry we aim to have when changing into canonical coordinates. We have this precisely when our equation is autonomous. There is a way to discover the most general ODE formula of a particular symmetry from ξ and η, however this is a process that requires more background and understanding and thus will not be included in the scope of this paper.

Example 5.4. *Consider the equation*

$$\frac{dy}{dx} = \frac{xy}{x^2 + y} \tag{5.8}$$

which can be rewritten as

$$\frac{dy}{dx} = \frac{y}{x + \frac{y}{x}}.$$

By looking at the above table, we can see that this equation is of the form

$$\frac{dy}{dx} = \frac{y}{x + F[\frac{y}{x}]}$$

thus we have $\xi = xy$ *and* $\eta = y^2$ *as the symmetry tangents. We can treat* $B = \frac{y}{x+\frac{y}{x}}$ *leaving* $A = 1$ *to calculate the integrating factor (5.7). Therefore*

$$M(x,y) = \frac{1}{1 \cdot y^2 - \frac{y}{x+\frac{y}{x}} \cdot xy} = \frac{x^2+y}{y^3}.$$

We can now simplify the equation $-M\,B\,dx + M\,A\,dy = 0$

$$\begin{aligned} 0 &= -\frac{x^2+y}{y^3} \cdot \frac{xy}{x^2+y}\,dx + \frac{x^2+y}{y^3} \cdot 1dy \\ &= -\frac{x}{y^2}dx + \left(\frac{x^2}{y^3} + \frac{1}{y^2}\right) dy \end{aligned}$$

and after using partial fractions and other standard integrating techniques, we get:

$$\begin{aligned} -\int \frac{x}{y^2}dx &= -\frac{x^2}{2y^3} + C_1(y) \\ \int \left(\frac{x^2}{y^3} + \frac{1}{y^2}\right) dy &= -\frac{1}{y} - \frac{x^2}{2y^2} + C_2(x) \end{aligned}$$

therefore the implicit solution to the ODE (5.8) is

$$-\frac{1}{y} - \frac{x^2}{2y^2} + C = 0.$$

Example 5.5. *Consider the ODE*

$$\frac{dy}{dx} = \frac{y}{x} + x\left(1 + \frac{y^2}{x^2}\right).$$

We can see that this is a function of the form

$$y' = \frac{y}{x} + xF\left[\frac{y}{x}\right]$$

so we have the tangent vector $(\xi(x,y), \eta(x,y)) = \left(1, \frac{y}{x}\right)$ *from the table. Again, for simplicity we will treat* $B = \frac{y}{x} + x\left(1 + \frac{y^2}{x^2}\right)$ *leaving* $A = 1$*. The integrating factor is:*

$$M = \frac{1}{1 \cdot \frac{y}{x} - \left(\frac{y}{x} + x\left(1 + \frac{y^2}{x^2}\right)\right) \cdot 1} = -\frac{x}{x^2+y^2}.$$

Thus we can impliment this into the Pfaffian form of the equation to get

$$\begin{aligned} 0 &= \frac{x}{x^2+y^2} \cdot \left(\frac{y}{x} + x \left(1 + \frac{y^2}{x^2} \right) \right) dx - \frac{x}{x^2+y^2} dy \\ &= \frac{x^2+y+y^2}{x^2+y^2} dx - \frac{x}{x^2+y^2} dy. \end{aligned}$$

The dx side of the equation solves to be:

$$\int \frac{x^2+y+y^2}{x^2+y^2} dx = -x - \arctan\left(\frac{x}{y}\right) + C_1(y)$$

and the dy side solves to be:

$$-\int \frac{x}{x^2+y^2} dy = -\arctan\left(\frac{y}{x}\right) + C_2(x).$$

Therefore our implicit solution is

$$-x - \arctan\left(\frac{x}{y}\right) + C = 0.$$

Bibliography

[C] Brian J. Cantwell, *Introduction to Symmetry Analysis,* Cambridge University Press, 2002. ISBN: 0-521-77183-8.

[E] C. H. Edwards, Jr., *Advanced Calculus of Several Variables,* Dover Publications Inc., 1995. ISBN: 0-486-68336-2.

[F] John B. Fraleigh, *A First Course in Abstract Algebra (7th Edition),* Pearson Education Inc., 2003. ISBN: 0-201-76390-7.

[FIS] Stephen H. Friedberg, Arnold J. Insel, Lawrence E. Spence, *Linear Algebra (4th Edition),* Prentice Hall India, 2002. ISBN:978-81-203-2606-4.

[H] Peter E. Hydon, *Symmetry Methods for Differential Equations,* Cambridge University Press, 2000. ISBN: 0-521-49703-5.

[I] Nail H. Ibragimov, *Elementary Lie Group Analysis and Ordinary Differential Equations,* John Wiley & Sons Ltd., 1999. ISBN: 0-471-97430-7.